SECTIONAL CURVATURES OF GRASSMANN MANIFOLDS

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(1) *Introduction.*—Let F be the field R of real numbers, the field C of complex numbers, or the field H of real quaternions; F^{n+m} a left $(n+m)$-dimensional Hermitian vector space over F; and $G_n(F^{n+m})$ the Grassmann manifold of n-planes in F^{n+m} provided with the invariant Riemannian metric with respect to which the distance between two points $\mathbf{A}$ and $\mathbf{B}$ in $G_n(F^{n+m})$ is equal to the square root of the sum of the squares of the angles between the n-planes $\mathbf{A}$ and $\mathbf{B}$ in F^{n+m} (see ref. 14).

Previous studies of $G_n(F^{n+m})$ have not unearthed sufficiently precise information about its sectional curvatures. Although the components of the curvature tensor at a point of $G_n(R^{n+m})$ and $G_n(C^{n+m})$ have been computed in references 5, 8, and 10, all that is so far known about the sectional curvatures of $G_n(F^{n+m})$ seems to be that they are nonnegative but not all positive unless $\min(n,m) = 1$, in which case they have the range of values given in Theorem 1 in §(4) (see ref. 2, p. 171; ref. 3, Theorem 7; ref. 1, p. 59; ref. 9, pp. 351 and 358; and ref. 8, Theorem 4.5).

In this note, which is a continuation of reference 14 and a companion to reference 15, we shall give a complete description of the sectional curvatures of $G_n(F^{n+m})$. In §§ (2) and (3), we express the curvature tensor and the sectional curvature of $G_n(F^{n+m})$ in terms of local coordinates and define the unitary curvature of $G_n(H^{n+m})$. In §(4), we state our main results concerning the range of values of the sectional curvature and certain characteristic properties of sections of minimum and maximum curvatures. Details of these results and similar results for the classical bounded symmetric domains will be published later.

(2) *The Curvature Tensor.*—We know (ref. 14) that the invariant Riemannian metric on $G_n(F^{n+m})$ can be arrived at in a natural and geometric way; moreover, in a typical local coordinate system (U,Z) with neighborhood U and coordinate Z (which is an $n \times m$ matrix with elements in F), it has the explicit expression

$$ds^2 = Re\ Tr[(I + ZZ^*)^{-1}dZ(I + Z^*Z)^{-1}dZ^*], \tag{1}$$

where Z^* is the conjugate transpose of Z, and $Re\ Tr$ denotes the real part of the trace. (For $F = C$, $n = 1$, and $m > 1$, (1) reduces to the Fubini metric of a complex projective space; see, for example, ref. 4, §(7).) If T_1,T_2 are any two tangent vectors at the point $Z \in U$ represented by their component matrices, then (1) is equivalent to

$$g_Z(T_1,T_2) = Re\ Tr[(I + ZZ^*)^{-1}T_1(I + Z^*Z)^{-1}T_2^*]. \tag{2}$$

To find the curvature tensor, we use the method of C. L. Siegel (ref. 11, §(17)) and L. K. Hua (ref. 7, §(8)). It is known (ref. 14) that in (U,Z) the differential equation of the geodesics is

$$\ddot{Z} - 2\dot{Z}Z^*(I + ZZ^*)^{-1}\dot{Z} = 0,$$

where the dots denote derivatives with respect to the arc length. Therefore, parallel displacement of vectors $T(t)$ along a curve segment $Z = Z(t), t \in [a,b] \subset R$, is characterized by

$$\frac{dT}{dt} = T\,Z^*(I + ZZ^*)^{-1}\frac{dZ}{dt} + \frac{dZ}{dt}Z^*(I + ZZ^*)^{-1}T. \tag{3}$$

From this and

$$R_Z(d_1Z,d_2Z)T = (d_2d_1 - d_1d_2)T,$$

we can derive the following expression for the curvature tensor R_Z of $G_n(F^{n+m})$ at Z:

$$\begin{aligned} R_Z(T_1,T_2)T = {} & T[(I + Z^*Z)^{-1}T_2^*(I + ZZ^*)^{-1}T_1 \\ & - (I + Z^*Z)^{-1}T_1^*(I + ZZ^*)^{-1}T_2] + [T_1(I + Z^*Z)^{-1}T_2^*(I + ZZ^*)^{-1} \\ & - T_2(I + Z^*Z)^{-1}T_1^*(I + ZZ^*)^{-1}]T. \end{aligned} \tag{4}$$

(3) *Sectional Curvature and Unitary Curvature.*—The curvature for the (real) plane section spanned by the tangent vectors T_1,T_2 at the point Z is defined as

$$K_Z(T_1,T_2) = -\frac{g_Z(R_Z(T_1,T_2)T_1,T_2)}{|T_1 \vee T_2|^2} \tag{5}$$

(see, for example, ref. 6, p. 65). Using (2) and (4) in (5), we obtain, for the tangent vectors T_1,T_2 at the point $Z = 0 \in U$,

$$K_0(T_1,T_2) = \frac{2\,Tr[(T_1T_2^* - T_2T_1^*)(\ )^* + (T_1^*T_2 - T_2^*T_1)(\ )^*]}{4\,Tr(T_1T_1^*)Tr(T_2T_2^*) - [Tr(T_1T_2^* + T_2T_1^*)]^2}, \tag{6}$$

where $(\ )^*$ denotes the conjugate transpose of the expression inside the preceding pair of brackets.

We can prove

LEMMA. (a) *Given any tangent vector T at the point $Z = 0 \in U$, there exists a unitary matrix N of order n such that*

$$N\zeta N^*TT^* = TT^*N\zeta N^* \quad \textit{for all } \zeta \in F.$$

(b) *For any such unitary matrix N and any nonreal $\zeta \in F$, we have*

$$K_0(T,N\zeta N^*T) = \frac{4\,Tr(TT^*TT^*)}{[Tr(TT^*)]^2}. \tag{7}$$

For a $G_n(F^{n+m})$, where $F = C$ or H, we call the real section spanned by T and any such $N\zeta N^*T$ a *unitary section* at $Z = 0$, and the function of T on the right side of equation (7) the *unitary curvature* for T at $Z = 0$. It is easy to see that, for a $G_n(C^{n+m})$, unitary section coincides with holomorphic section and unitary curvature coincides with holomorphic curvature (ref. 3, §(2)). As in reference 14, §(11), we can show that *for a* $G_n(F^{n+m})$, *where* $F = C$ *or* H, *the range of values of the unitary curvature* k_Z *is*

$$\frac{4}{\min(n,m)} \leqslant k_Z(T) \leqslant 4.$$

(4) *Range of Values of the Sectional Curvature and Sections of Minimum and Maximum Curvatures.*—A few preliminary remarks must precede the statement of our main results. We know (ref. 14, Theorem 13) that any Grassmann manifold can be naturally, isometrically imbedded in a "larger" Grassmann manifold as a totally geodesic submanifold. In particular, FP^1, RP^2, and RP^m can all be imbedded in a $G_n(F^{n+m})$ in this manner if $\min(n,m) \geqslant 2$. We know also (ref. 13, Theorems 7.2(i) and 8.1; and ref. 12, Theorem 2) that in a Euclidean R^4, any maximal family of mutually isoclinic 2-planes, when viewed as a subset of $G_2(R^4)$, is a totally geodesic submanifold Φ of $G_2(R^4)$, which is isometric with a 2-dimensional sphere of radius $1/\sqrt{2}$. When $G_2(R^4)$ is naturally imbedded in $G_n(R^{n+m})$, where $\min(n,m) \geqslant 2$, the image of Φ is a totally geodesic submanifold of $G_n(R^{n+m})$ which we call a *geodesic* 2-*sphere* in $G_n(R^{n+m})$.

We can now state our main results. The proof which we omit is based on formula (6), matrix algebra over F, and our knowledge of $G_n(F^{n+m})$ obtained by the method described in reference 14.

THEOREM 1. (Known; see §(1), second paragraph.) *Let us denote* $G_1(F^{1+m})$ *by* FP^m. *Then*

(a) *For a* RP^1, $K_Z(T_1,T_2)$ *is not defined. For a* RP^m, *where* $m \geqslant 2$, $K_Z(T_1,T_2) = 1$.

(b) *For a* CP^1 *or* HP^1, $K_Z(T_1,T_2) = 4$. *For a* CP^m *or* HP^m, *where* $m \geqslant 2$, $1 \leqslant K_Z(T_1,T_2) \leqslant 4$.

THEOREM 2. *For an* $FP^m = G_1(F^{1+m})$, *where* $F = C$ *or* H *and* $m \geqslant 2$, *the following are true:*

(a)(i) *A section at a point of* FP^m *has minimum curvature* 1 *iff the geodesics tangent to it generate a totally geodesic submanifold of real dimension* 2 *which is isometric with an* RP^2.

(ii) *A set* Ψ *of tangent vectors at a point* Z *is a maximal set such that* $K_Z(T_1,T_2) =$ *minimum* 1 *for every* $T_1,T_2 \in \Psi$ *iff the geodesics tangent to the vectors of* Ψ *at* Z *generate a totally geodesic submanifold of real dimension* m *which is isometric with an* RP^m.

(iii) *Given any tangent vector* T *at a point* Z, *there exists at least one such maximal set* Ψ *containing* T.

(b) *A section has maximum curvature* 4 *iff it is a unitary section.* (For CP^m, (*b*) is known; see ref. 3, Theorem 7.)

THEOREM 3. (a) *For a* $G_n(R^{n+m})$, *where* $\min(n,m) \geqslant 2$,

$$0 \leqslant K_Z(T_1,T_2) \leqslant 2.$$

(b) *For a* $G_n(C^{n+m})$ *or a* $G_n(H^{n+m})$, *where* $min(n,m) \geqslant 2$,

$$0 \leqslant K_Z(T_1,T_2) \leqslant 4.$$

THEOREM 4. *For a* $G_n(F^{n+m})$, *where* $min(n,m) \geqslant 2$, *the following are true:*

(a) $K_Z(T_1,T_2)$ = *minimum* 0 *iff the two geodesics tangent to* T_1,T_2 *at* Z *have a common frame of mutually orthogonal angle-planes* (see ref. 14, Theorem 7).

(b) *A maximal set* Ψ *of tangent vectors at a point* Z *such that* $K_Z(T_1, T_2)$ = *minimum* 0 *for every* $T_1,T_2 \in \Psi$ *is characterized by the property that the geodesics tangent to the tangent vectors in* Ψ *at* Z *generate a totally geodesic submanifold of* $G_n(F^{n+m})$ *which is isometric with a flat torus of real-dimension* $min(n,m)$.

(c) *At each point of* $G_n(F^{n+m})$, *there are* ∞^{nm-r}, ∞^{2nm}, or ∞^{4nm+2r} *such maximal sets* Ψ *according as* $F = R, C$ *or* H, *where* $r = min(n,m)$.

An easy consequence of Theorem 4(b) is the result, already known, that the maximum dimension of a flat, totally geodesic submanifold of $G_n(F^{n+m})$ is $\min(n,m)$; in other words, the rank of $G_n(F^{n+m})$ is $\min(n,m)$ (see, for example, ref. 6, pp. 209, 349, and 351).

THEOREM 5. (a) *For a* $G_n(R^{n+m})$, *where* $min(n,m) \geqslant 2$, *the following are true:*

(i) *A section has maximum curvature* 2 *iff it is tangent to a geodesic* 2-*sphere.*

(ii) *Any geodesic tangent to a section of maximum curvature* 2 *is a closed geodesic of length* $\sqrt{2}\,\pi$.

(ii) *If* γ *is any closed geodesic of length* $\sqrt{2}\,\pi$ *and* Z *any point of* γ, *then there is a unique section of maximum curvature* 2 *tangent to* γ *at* Z.

(b) *For a* $G_n(F^{n+m})$, *where* $F = C$ *or* H *and* $min(n,m) \geqslant 2$, *a section has maximum curvature* 4 *iff it is tangent to a totally geodesic submanifold of* $G_n(F^{n+m})$ *which is isometric with an* FP^1; *or equivalently, iff it is a unitary section determined by a tangent vector to a closed geodesic of length* π. *Moreover, at each point, there are infinitely many sections of maximum curvature* 4.

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